## XV. *A new Theory of the Rotatory Motion of Bodies affected by Forces disturbing such Motion.* By Mr. John Landen, *F. R. S.*

Read Feb. 20, 1777.

I AM induced to consider this paper as not unworthy the notice of this Society, through a persuasion that the theory herein contained will conduce to the improvement of science, by enabling the reader to form a true idea, and accordingly to make a computation of the motion (or change) of the axis about which a body having a rotatory motion will turn, or have a tendency to turn, upon being affected by a force disturbing its rotation; particularly of the motion of the earth's axis arising from the attraction of the Sun and Moon on the protuberant matter of the earth above its greatest inscribed sphere: which compound motion, I conceive, has not been rightly explained by any one of the eminent mathematicians whose writings on the same subject have come to my hands. Whether in this essay I have really succeeded better than other writers who have attempted an explanation of such motion, I submit to gentlemen well versed in mechanics to determine.

1. Fig.

1. Fig. 1. Let the ſphere ADBE, whoſe radius is $r$, revolve uniformly about the diameter ACB as an axis, with the angular velocity $c$, meaſured at D or E, the motion being according to the order of the letters DGEH in the ſection at right angles to ACB, fig. 2.; and, whilſt it is ſo revolving, let the pole A be impelled by ſome inſtantaneous percuſſive force to turn about the diameter DCE, from A towards H, with the velocity $w$. It is propoſed to find the new axis about which the ſphere will revolve after receiving ſuch impulſe.

Calling $al$, parallel to DC, $x$; $cl$ will be $=\sqrt{r^2-x^2}$: the velocity of the point $a$ (about ACB) before the impulſe on A will be $=\frac{cx}{r}$; and the velocity (about DCE) given to the ſame point ($a$) by the ſaid impulſe will be $=\frac{w\sqrt{r^2-x^2}}{r}$. Which velocities of the point $a$ being in contrary directions, if it be ſo ſituated that they be equal, then, one deſtroying the other, that point will ſtop and become one of the new poles ſought, about which the former poles A and B will revolve with the velocity $w$; and the points D and E will revolve with the ſame velocity ($c$) as before the perturbating impulſe on the point A; but inſtead of deſcribing the great circle DGEH, their motion will be about the new axis $ab$; about which they (as well as the points A and B) will deſcribe leſſer circles parallel to the great circle *de*, in which the points *d* and *e* (*de* being at right angles to *ab*) will revolve about the ſame axis (*ab*) with the velocity $\sqrt{c^2+w^2}$. Which being denoted by *e*, and *m* and *n* being put for the ſine and coſine of the angle AC*a* to the radius 1, $me$ will be $=w$, $ne=c$, and conſequently $mne^2=cw$.

Now taking $\frac{cx}{r}=\frac{w\sqrt{r^2-x^2}}{r}$, in order to find that new axis *ab*, we have from that equation $x=\frac{rw}{\sqrt{c^2+w^2}}=\frac{rw}{e}=al$.

Moreover it is obvious, that if a ſpheroid, a cylinder, or any other body, whoſe center of gravity is C and proper axis ACB, were, whilſt revolving about that axis with the ſame angular velocity ($c$), to receive ſuch an impulſe as inſtantly to give the point A the angular velocity $w$ about DCE; the axis about which that ſpheroid, cylinder, or other body, immediately after the impulſe, would revolve, or would have a tendency to revolve, would be the ſame line *ab*.

The great circle *de* (fig. 1.) and any other great circle ſo ſituated with reſpect to the axis of any revolving ſphere, I ſhall denominate the *mid-circle*.

2. In the manner above deſcribed the poles of the ſphere are by the inſtantaneous impulſe on the point A inſtantly changed from A and B to *a* and *b*. But if, inſtead of ſuch impulſe, a continued attractive force F (like that of gravity)

vity) acted at A fig. 3. and at the new poles $\acute{a}$, $\acute{\acute{a}}$, &c. as they become fuch by a fucceffive change caufed by fuch continued action of the force F urging the fphere at every inftant to revolve about the diameter $\acute{d}\acute{e}$, or $\acute{\acute{d}}\acute{\acute{e}}$, &c. of the contemporary mid-circle, the new pole ($\acute{a}$, $\acute{\acute{a}}$, &c.) would not inftantly be at a finite diftance from the primitive pole A, but fome finite time would be requifite, that by fuch fucceffive change, the pole might be varied to a finite diftance from A: and the force F continuing invariable, the velocity ($v$) wherewith the pole changed its place would be expreffed by $\frac{z}{t}$, $t$ being the time elapfed whilft the pole is varying from A to $a$, and $z$ the length of the arc A $a$. Therefore the velocity wherewith the pole will change its place during fuch action of the force F will be expreffed in the fame manner as the velocity ($v$) of a body moving uniformly from A to $a$ in the time $t$ may be expreffed; that is, in both cafes $v$ will be $= \frac{z}{t}$.. But there is a material difference between the motion of a body fo moving from A to $a$ and the change of place of the pole $\acute{a}$, $\acute{\acute{a}}$, &c. the former is permanent, and will continue to carry the body forward without the action of any force whatever; whereas the latter will inftantly ceafe, and the axis will keep its pofition, if the force

force F ceafes to act thereon; like as the varying direction of a projectile near the earth's furface would immediately ceafe to change, if the force of gravity ceafed to act.

It is obfervable, that whilft the force F acts, and the revolving fphere, in confequence of fuch action, every moment takes a new axis, the angular motion about the axis will continue invariable; the action of fuch force only altering the axis without altering the angular velocity of the fphere about it: like as the direction of a moving body is altered, without altering the velocity thereof, by an attractive force continually acting on it in a direction at right angles to that in which the body is moving. And if ever the force F fhall ceafe to act, the fphere will inftantly revolve with its primitive velocity ($c$) about the axis it then may have been brought to take by the preaction of that force.

The new axis, about which the fphere has fuch tendency to revolve at any inftant during the action of the force F, I fhall call the *momentary axis*; and the poles thereof the *momentary poles*.

3. From the equation $\frac{cx}{r}=\frac{w\sqrt{r^2-x^2}}{r}$ (art. 1.) we have $\frac{w}{x}=\frac{c}{\sqrt{r^2-x^2}}$. Now if a continued attractive force (F) act during the time $t$ as above-mentioned, inftead of the

inſtantaneous percuſſive force at A, we, according to the doctrine of fluxions, muſt, inſtead of $w$, take $\dot{w}$, or its equal $F\dot{t}$, and $\dot{x}$ inſtead of $x$, in the expreſſion $\frac{w}{x}$; therefore, in this caſe, we have $\frac{\dot{w}}{\dot{x}} = \frac{F\dot{t}}{\dot{x}} = \frac{c}{\sqrt{r^2-x^2}}$. Whence, putting $z$ for the arc ($Aa'$, or $Aa''$, &c.) whoſe ſine is $x$, and writing $\dot{z}$ for its equal $\frac{r\dot{x}}{\sqrt{r^2-x^2}}$, we get $\frac{rF\dot{t}}{\dot{z}} = c$, or $\dot{z} = \frac{rF\dot{t}}{c}$.

Hence $v$ denoting the velocity wherewith the momentary pole ($a'$, $a''$, &c.) changes its place during the action of the accelerative force F, we have $\dot{z} = v\dot{t} = \frac{rF\dot{t}}{c}$, and conſequently $v = \frac{rF}{c}$

4. The value of $v$ may alſo be determined in the following manner (fig. 4.). Conceive a very thin ſtring (without weight) to have one of its ends faſtened to a fixed point $l$ and the other to a heavy particle of matter $m$; alſo conceive ſuch particle ſo to revolve with the velocity $e$, about the line $ln$, that a certain accelerative force F (like that of gravity referred to a certain direction) continually acting on the ſaid particle $m$, in a direction at right angles both to the ſtring $lm$ and to the tangent to the curve in which $m$ is moving, the ſtring ſhall deſcribe a conical ſurface. Then $lm$ being denoted by

by $r$, and $mo$, perpendicular to $ln$, by $q$; $\frac{e^2}{q}$, the centrifugal force urging $m$ in the direction $om$, will be to F as $r$ to $\sqrt{r^2-q^2}=lo$. Therefore F muſt be $=\frac{e^2\sqrt{r^2-q^2}}{rq}$. Now if, whilſt $m$ is ſo revolving, the force F ceaſes acting, the ſaid particle ($m$) will, it is obvious, immediately proceed to deſcribe a great circle of the ſphere whoſe radius is $r$ and center $l$, of which great circle one of the poles will be ſituated in a leſſer circle parallel to, and 90° diſtant from, that deſcribed by $m$ during ſuch action of the ſaid force; which pole, during ſuch action, will change its place in the ſaid leſſer circle in which it will at any time be found with a velocity ($v$) which will be to $e$ as ($s$) the radius of the laſt mentioned circle to $q$. But $s$ will be $=\sqrt{r^2-q^2}$; therefore we have $v:e::\sqrt{r^2-q^2}:q$, and $\frac{\sqrt{r^2-q^2}}{q}=\frac{v}{e}$. Conſequently $F=\frac{e^2\sqrt{r^2-q^2}}{rq}$ will be $=\frac{e^2}{r}\times\frac{v}{e}=\frac{ev}{r}$, and $v=\frac{rF}{e}$.

Let now $m$ be a point on the ſurface of a ſphere whoſe center is $l$, and radius $lm=r$; and let the ſphere revolve about an axis ſo that $m$ ſhall deſcribe a great circle with the velocity $e$. If then ſuch a motive force begins to act on the ſphere, that, continuing its action, the point $m$ ſhall always be urged by the invariable accelerative force F to move in a direction at right angles to the ray $lm$ and to the tangent to the curve which $m$ will defcribe; that point it is obvious will, in confequence of the action of that force, defcribe a leffer circle of the fame radius ($q$) as that defcribed by the particle $m$ when faftened to a ftring and acted on by the force F as above-mentioned; and the center of the fphere being always confidered as at reft, one of the momentary poles of the fphere will defcribe a circle whofe radius will be $= \sqrt{r^2 - q^2}$ parallel to, and 90° diftant from, that defcribed by the point $m$. For if the faid force were to ceafe acting, that point of the fphere would defcribe a great circle, as would the particle $m$ at the ftring in the like cafe; and therefore both the faid particle and the point $m$ of the fphere at every inftant having the fame tendency, and being acted on by equal accelerative forces, the effect will be the fame with refpect to the motion of each. Confequently, $v$ being put to denote the velocity wherewith the momentary pole changes its place in the circle which it will defcribe whilft the motive force producing the accelerative force F acts on $m$ as juft now mentioned, $v$ will be $= \frac{rF}{e}$, the fame as in the preceding article, $e$ here denoting that velocity which we there denoted by $c$.

5. Referring the point of action of the perturbating force to the mid-circle we have not hitherto confidered

that point as varied with a greater or leſs velocity than ($e$) that of the point $m$; that is, with reference to ſuch circle we have always conſidered the point $m$ as the point of action. But it is obvious, that, *cæteris paribus*, the point of action with reſpect to the mid-circle (which point we will now denote by $q$) may be varied with a velocity greater or leſs than $e$; and that, *cæteris paribus*, the velocity ($v$) of the momentary pole will be the ſame with what velocity ſoever ($q$) the point of action of the force F be varied; the direction in which that force acts being always at right angles to the ray ($lq$) from the center of the ſphere, and to the tangent to the curve deſcribed by ($q$) ſuch point of action.

Yet, although $v$ continues the ſame whether, *cæteris paribus*, ($u$) the velocity of the point $q$ be greater, equal to, or leſs than $e$, the immoveable circle in which the momentary pole will be found will not continue the ſame; that circle being greater, equal to, or leſs than the circle whoſe radius is $\sqrt{r^2-q^2}$ according as $u$ is leſs, equal to, or greater than $e$, as will be made more evident by what follows.

6. Fig. 5. Let $\acute{p}$ (in the great circle R$\acute{p}$Q$\acute{q}$T) be one of the poles of the axis about which the ſphere RSTV, whoſe radius is $r$, is revolving (according to the order of the letters V$\acute{q}$S) with the angular velocity $e$, meaſured at

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the diſtance $r$ from the axis; and whilſt it is ſo revolving let the ſaid pole be urged to turn about a diameter of the mid-circle V$q$S towards $q$, by an accelerative force F, and let ſuch force continue to act on the ſucceſſive new poles $p''$, $p'''$, &c. as they become ſuch, always urging the ſphere to turn about a diameter of the contemporary mid-circle, whilſt the direction in which ſuch perturbating force acts is regulated in the following manner.

Conceive the ſaid revolving ſphere to be ſurrounded by an immoveable concave ſphere of the ſame radius $r$. Then the momentary pole ($p'$, $p''$, $p'''$, &c.) will always be found in ſome curve $P'P''P'''$ &c. in the ſaid concave ſphere, and in ſome curve $p'p''p'''$ &c. on the revolving ſphere; which laſt mentioned curve will continually touch and roll along the other curve $P'P''P'''$ &c. on the immoveable ſphere, the force F and the direction in which it acts varying in any manner whatever. Let F be invariable; then, it is obvious, the two curves ſo touching each other will be circles; and if great circles $P'q'$, $P''q''$, $P'''q'''$, &c. be deſcribed on the ſurface of the immoveable ſphere whoſe planes ſhall be at right angles to the plane of the circle $P'P''P'''$ &c. the points $q'$, $q''$, $q'''$, &c.

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therein, each 90° distant from $\overset{\prime}{P}$, $\overset{\prime\prime}{P}$, $\overset{\prime\prime\prime}{P}$, &c. respectively, will be in a circle ($\overset{\prime}{q}\overset{\prime\prime}{q}\overset{\prime\prime\prime}{q}$ &c.) parallel to the said circle $\overset{\prime}{P}\overset{\prime\prime}{P}\overset{\prime\prime\prime}{P}$ &c. Now as a regulation to the direction in which the force F shall urge the momentary pole, let that direction be always a tangent to the great circle so passing through that pole and the correspondent point $\overset{\prime\prime}{q}$, or $\overset{\prime\prime\prime}{q}$, &c. whilst the arcs $\overset{\prime}{q}\overset{\prime\prime}{q}$, $\overset{\prime}{q}\overset{\prime\prime\prime}{q}$, &c. are to the arcs $\overset{\prime}{P}\overset{\prime\prime}{P}$, $\overset{\prime}{P}\overset{\prime\prime\prime}{P}$, &c. respectively in the constant ratio of $u$ to $v$.

The direction in which the force F acts being so regulated, it is obvious that the radius of the circle $\overset{\prime}{P}\overset{\prime\prime}{P}\overset{\prime\prime\prime}{P}$ &c. being denoted by $b$, the radius of the circle $\overset{\prime}{q}\overset{\prime\prime}{q}\overset{\prime\prime\prime}{q}$ &c. will be $=\sqrt{r^2-b^2}$, the distance of these parallel circles being 90°. Therefore their peripheries being as the velocities ($v$ and $u$) with which they are described, their radii ($b$ and $\sqrt{r^2-b^2}$) will be in the ratio of the said velocities; that is $v:u::b:\sqrt{r^2-b^2}$; whence, $\frac{u}{v}$ being $=\frac{\sqrt{r^2-b^2}}{b}$, $b$, the radius of the circle $\overset{\prime}{P}\overset{\prime\prime}{P}\overset{\prime\prime\prime}{P}$ &c. is found $=\frac{rv}{\sqrt{u^2+v^2}}$ $=\frac{r}{\sqrt{1+\frac{e^2u^2}{r^2F^2}}}$; and $\sqrt{r^2-b^2}$ the radius of the circle $\overset{\prime}{q}\overset{\prime\prime}{q}\overset{\prime\prime\prime}{q}$ &c. $=\frac{ru}{\sqrt{u^2+v^2}}=\frac{r}{\sqrt{1+\frac{r^2F^2}{e^2u^2}}}$, $v$ being $=\frac{rF}{e}$, the velocity where-

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with the momentary pole $\overset{\prime\prime}{p}$, $\overset{\prime\prime\prime}{p}$, &c. changes its place.

Consequently, if $\overset{\prime}{\text{P}}\text{R}$ be an arc in the said immoveable concave sphere whose sine is $\frac{rv}{\sqrt{u^2+v^2}} = \frac{r}{\sqrt{1+\frac{e^2u^2}{r^2F^2}}}$, the great circles $\overset{\prime}{q}\overset{\prime}{\text{P}}$, $\overset{\prime\prime}{q}\overset{\prime\prime}{\text{P}}$, $\overset{\prime\prime\prime}{q}\overset{\prime\prime\prime}{\text{P}}$, &c. will interfect each other at the point R.

7. Moreover, the force F being invariable and acting as expressed in the preceding article, the primitive pole $\overset{\prime}{p}$ and the momentary poles $\overset{\prime\prime}{p}$, $\overset{\prime\prime\prime}{p}$, &c. will all be found in a circle $\overset{\prime}{p}\overset{\prime\prime}{p}\overset{\prime\prime\prime}{p}$ &c. described upon the surface of the revolving sphere, as observed in that article; which circle, during the action of the force of F, will (as is also observed in the said article) always touch and roll along the immoveable circle ($\overset{\prime}{\text{P}}\overset{\prime\prime}{\text{P}}\overset{\prime\prime\prime}{\text{P}}$ &c.) whose radius we have just now found $= \frac{rv}{\sqrt{u^2+v^2}} = \frac{r}{\sqrt{1+\frac{e^2u^3}{r^2F^2}}}$; the point of contact being always the momentary pole.

Let the sine of the arc $\overset{\prime}{\text{P}}\text{Q}$ of the great circle $\text{R}\overset{\prime}{p}\overset{\prime}{\text{Q}}\overset{\prime}{q}\text{T}$ in the revolving sphere be equal to $k$, the radius of the said circle $\overset{\prime}{p}\overset{\prime\prime}{p}\overset{\prime\prime\prime}{p}$ &c. then will the point Q and its opposite point (o) in the surface of the said sphere, during the action of the force F, describe circles in the sur-

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rounding immoveable concave ſphere parallel to ($\acute{P}\, \overset{\prime\prime}{P}\, \overset{\prime\prime\prime}{P}$ &c.) the circle deſcribed by the momentary pole $\overset{\prime\prime}{p}$, $\overset{\prime\prime\prime}{p}$, &c. in the ſame concave ſphere. And ſuch point Q and its oppoſite point (o) being continually urged by the force F in directions at right angles to the tangents to the arcs they deſcribe, their velocity will continue the ſame as before the action of the ſaid force commenced; which velocity, and the radius of the ſaid circle $\acute{p}\,\overset{\prime\prime}{p}\,\overset{\prime\prime\prime}{p}$ &c. will be determined by the following computation.

That radius being denoted by $k$, we have $r : k :: e : \frac{ek}{r}$, the velocity of the point Q before the action of the force F commenced; and $b : v :: K : \frac{Kv}{b}$, the velocity of the ſame point (Q) during the action of that force, K being put for the ſine of the arc QR; therefore the velocity of Q continuing the ſame during the action of F as before, we have $\frac{ek}{r} = \frac{Kv}{b}$. But K is the ſine of the ſum of the arcs $R\acute{P}$, $\acute{P}Q$, whoſe ſines are $b$ and $k$ reſpectively; therefore $\frac{b\sqrt{r^2-k^2}}{r} + \frac{k\sqrt{r^2-b^2}}{r}$ will be $=$ K; and by ſubſtitution we get $\frac{ek}{r} = \frac{v\sqrt{r^2-k^2}}{r} + \frac{kv\sqrt{r^2-b^2}}{r} = \frac{v\sqrt{r^2-k^2}}{r} + \frac{ku}{r}$, $\frac{\sqrt{r^2-b^2}}{b}$ being $= \frac{u}{v}$ by the preceding article. Hence we find $k =$

$\frac{rv}{\sqrt{\overline{e-u}|^2+v^2}}$; and it follows, that $\frac{ev}{\sqrt{\overline{e-u}|^2+v^2}}$ $(=\frac{ek}{r})$ will be equal to the velocity of the point Q, and likewiſe of its oppoſite point (o) in the ſurface of the ſphere. It alſo follows, that K, the radius of each of the circles deſcribed by thoſe points, during the action of the force F will be equal to $\frac{rev}{\sqrt{u^2+v^2}\times\sqrt{\overline{e-u}|^2+v^2}}$.

By what is done it appears, that during the action of the force F, the motion of the revolving ſphere will be regulated by the circle $\overset{\prime}{p}\overset{\prime\prime}{p}\overset{\prime\prime\prime}{p}$ &c. thereon (whoſe radius is $\frac{rv}{\sqrt{\overline{e-u}|^2+v^2}}=\frac{r}{\sqrt{1+\frac{e^2\overline{e-u}|^2}{r^2F^2}}}$) continually touching and rolling along the immoveable circle $\overset{\prime}{P}\overset{\prime\prime}{P}\overset{\prime\prime\prime}{P}$ &c. (whoſe radius is $\frac{rv}{\sqrt{u^2+v^2}}=\frac{r}{\sqrt{1+\frac{e^2u^2}{r^2F^2}}}$) ſo that the velocity of the point of contract be $=v=\frac{rF}{e}$. Conſidering the point Q as always urged from the points $\overset{\prime}{P}$, $\overset{\prime\prime}{P}$, $\overset{\prime\prime\prime}{P}$, &c. and conſequently its oppoſite point (o) towards thoſe points, it is neceſſary to obſerve, that according as $u$ is leſs or greater than $e$, the arc PQ (whoſe ſine is $\frac{rv}{\sqrt{\overline{e-u}|^2+v^2}}$) will be leſs or greater than 90°; and the point (o) oppoſite to Q on the ſurface of the ſphere will accordingly be at a greater or leſs diſtance than 90° from P.

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If $u$ be negative the arc PR whoſe ſine is $\frac{rv}{\sqrt{u^2+v^2}}$ will be greater than 90°.

8. The motion of the ſphere according to the regulation in the preceding article is one motion compounded of the primitive motion of the ſphere and the motion generated by the action of the force F. But conceiving $\left(\frac{ev}{\sqrt{\overline{e-u}|^2+v^2}}\right)$ the velocity of the point Q to ariſe from an impulſe given to it whilſt the ſphere revolved about an axis of which Q was an immoveable pole before ſuch impulſe, and about which the mid-circle correſponding to that primitive axis revolved with the angular velocity $\frac{e.\overline{e \backsim u}}{\sqrt{\overline{e-u}|^2+v^2}}$ (a); and conſidering that the force F, continually acting at right angles to the momentary direction of the point Q and to the plane of the ſaid mid-circle, only ſerves to alter the poſition of the ſaid primitive axis; we may, by the help of what is done above, explain the motion which the ſphere will have, during the action of the force F, ſo as to retain in our ideas the two primitive motions (one about the axis QO, and the other about a diameter at right angles to that axis) as remaining diſtinct and unaltered.

(a) Denoting this by $c$ and the velocity of Q by $d$, $\sqrt{c^2+d^2}$ is $= e$, agreeable to art. 1.

Fig.

Fig. 6. Let ED be a great circle on the revolving ſphere, of which Q is a pole, and let a ſmaller circle DL parallel to (MQ) that which we have found will be deſcribed by the point Q, be drawn on the immoveable concave ſphere ſo as to touch that great circle in the point (D) where the great circle QPR cuts it; the radius of which leſſer circle will be $(=\sqrt{r^2-K^2}=)\frac{rv^2 \backsim ru.\overline{e-u}}{\sqrt{u^2+v^2}\times\sqrt{\overline{e-u}|^2+v^2}}$.

Then the revolving ſphere, during the action of the force F, will ſo move, that the firſt mentioned great circle (ED) ſhall continually touch and roll along the ſaid leſſer circle DL, the velocity of the point of contact (along that circle) being $=\frac{v^2 \backsim u.\overline{e-u}}{\sqrt{\overline{e-u}|^2+v^2}}$ (*b*), and the ſphere at the ſame time turning about the axis of which Q is a pole with the primitive angular velocity $\frac{\overline{e.e \backsim u}}{\sqrt{\overline{e-u}|^2+v^2}}$.

Thus the primitive motion about the axis of which Q is a pole is preſerved diſtinct, whilſt that pole proceeds deſcribing a circle, whoſe radius is $\frac{rev}{\sqrt{u^2+v^2}\times\sqrt{\overline{e-u}|^2+v^2}}$, with the velocity $\frac{ev}{\sqrt{\overline{e-u}|^2+v^2}}$ which we ſuppoſed given to it.

(*b*) This is to the velocity of the point Q as $\sqrt{r^2-K^2}$ to K; that is, as the radii of the arcs deſcribed.

It is obfervable, that the laft mentioned velocity will, according to this regulation of the motion, be to the primitive angular velocity about the axis of which Q is a pole, as $v$ to $e-u$, or as $v$ to $u-e$, according as $u$ is lefs or greater than $e$; that is, according as the arc PQ is lefs or greater than 90°.

9. From what has been faid it follows, that denoting the two primitive angular velocities $\frac{e.e \backsim u}{\sqrt{\overline{e-u}^2+v^2}}$ and $\frac{ev}{\sqrt{\overline{e-u}^2+v^2}}$ (fpecified in the preceding article) by $c$ and $d$ refpectively, the radius (fig. 5.) of the circle $p'p''p'''$ &c. (or fine of the arc $PQ'=PQ''$, &c.) will be $=\frac{dr}{e}$; the radius of the circle $P'P''P'''$ &c. (or fine of the arc $PR'=PR''$, &c.) $=\frac{dr^2F}{e\sqrt{d^2e^2\mp 2cdrF+r^2F^2}}$: a great circle paffing through the primitive poles O and Q, on the revolving fphere, will turn from the pofition ORQ with the velocity $\frac{rF}{d}$ meafured at the mid-circle, or with the velocity $\frac{drF}{\sqrt{d^2e^2\mp 2cdrF+r^2F^2}}$ meafured at the fixed point R; whilft thofe poles defcribe, with the velocity $d$, circles parallel to $P'P''P'''$ &c. the radius (K) of each of the circles (fig. 6.) fo defcribed

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being $= \frac{d^2 r}{\sqrt{d^2 e^2 \mp 2cdr\text{F} + r^2\text{F}^2}}$: the radius $(\sqrt{r^2 - \text{K}^2})$ of the circle DL will be $= \frac{r \times \overline{cd \overset{\text{co}}{+} r\text{F}}}{\sqrt{d^2 e^2 \mp 2edr\text{F} + r^2\text{F}^2}}$; and the velocity $\left(\frac{v^2 \text{co}\, e.\, \overline{e - u}}{\sqrt{\overline{e-u}|^2 + v^2}}\right)$ along the ſaid circle DL $= c \overset{\text{co}}{+} \frac{r\text{F}}{d}$: the upper or lower of the double ſigns taking place according as $u$ $(= e \mp \frac{cr\text{F}}{de})$ is leſs or greater than $e$; that is, according as the arc $\text{P}'\text{Q}$ (whoſe ſine is $= \frac{dr}{e}$) is leſs or greater than 90°.

10. As an inſtance of the uſe of the preceding concluſions, I will now apply them in the ſolution of a very intereſting problem, which I have not before ſeen ſolved.

*Suppoſe a given ſpheroid, whilſt revolving uniformly about its proper axis, with a given angular velocity, to be ſuddenly urged by ſome percuſſive force to turn, with ſome given angular velocity, about a diameter of its equator; it is propoſed to explain the rotatory motion of the ſpheroid conſequent to the impulſe ſo received.*

Fig. 7, 8. Let DOEQ be the ſpheroid, whoſe ſemi-axis CO = CQ is $= b$, and equatorial radius CD = CE $= r$; and ſuppoſing it before the impulſe to revolve about its proper axis OQ with the angular velocity $c$, meaſured at the diſtance $r$ from the axis, let the poles (O and Q) be ſuddenly urged by ſome percuſſive force to turn about a

 diameter

diameter of the equator of the ſpheroid, with the angular velocity $d$, likewiſe meaſured at the diſtance $r$ from that diameter. Upon receiving ſuch impulſe, the ſpheroid will take a new axis of motion, which will be a momentary one; ſuppoſe ſuch new axis to be $p$ C $\pi$ (c). Then the particles of the ſpheroid being urged (or having a tendency) to turn about that axis with the angular velocity $\sqrt{c^2+d^2}$, (which we will denote by $e$) their joint centrifugal force will ſo urge the ſpheroid to turn about that diameter of the equator which ſhall be at right angles to the momentary axis $p$ C $\pi$, that the accelerative force of the point D of the equator to turn it about the ſaid diameter according to the order of the letters D Q E will (as appears by what is proved in art. 1. and in the Appendix annexed hereto) be $= \frac{cd}{r} \times \frac{r^2-b^2}{r^2+b^2}$ or $\frac{cd}{r} \times \frac{b^2-r^2}{r^2+b^2}$ according as $b$ is leſs or greater than $r$: and it follows from hence and what is proved in art. 3. and 4. that $v$, the angular velocity (at the diſtance $r$ from C) with which the momentary pole $p$ will change its place, will accordingly be $=$ $\frac{cd}{e} \times \frac{r^2-b^2}{r^2+b^2}$ or $\frac{cd}{e} \times \frac{b^2-r^2}{r^2+b^2}$.

(c) To find the poſition of this axis ſee art. 1. by which the ſine of the angle O C $p$ (to the radius $r$) is found $= \frac{dr}{e}$.

More-

Moreover, referring to our obſervation in art. 8. let $u-e$ be to $\frac{cd}{e} \times \frac{r^2-b^2}{r^2+b^2}$ (the value of $v$) as $c$ to $d$, $u$ being greater than $e$; or let $e-u$ be to $\frac{cd}{e} \times \frac{b^2-r^2}{r^2+b^2}$ as $c$ to $d$, $u$ being leſs than $e$: whence, in both caſes, we ſhall have the ſame expreſſion $\left(\frac{c^2}{e} \times \frac{r^2-b^2}{r^2+b^2}\right)$ for the value of $u-e$; and conſequently $u$, in both caſes, will be $= e + \frac{c^2}{e} \times \frac{r^2-b^2}{r^2+b^2}$.

Conceive now a ſpherical ſurface without matter, having the ſame center and radius as the equator DE, to be carried about with the revolving ſpheroid; and ſuppoſe a ſphere, whoſe radius is $r$, to revolve about an axis $pc\pi$ with the angular velocity $e$, and, whilſt it is ſo revolving, let an accelerative force (F) equal to $\frac{cd}{r} \times \frac{r^2-b^2}{r^2+b^2}$ or $\frac{cd}{r} \times \frac{b^2-r^2}{r^2+b^2}$, according as $b$ is leſs or greater than $r$, urge the pole $p$, and the ſucceſſive momentary poles as they become ſuch, to turn about a diameter of the contemporary mid-circle in the manner expreſſed in art. 6. $u$ being to $v$ as $e + \frac{c^2}{e} \times \frac{r^2-b^2}{r^2+b^2}$ to $\frac{cd}{e} \times \frac{r^2-b^2}{r^2+b^2}$ or as $e + \frac{c^2}{e} \times \frac{r^2-b^2}{r^2+b^2}$ to $\frac{cd}{e} \times \frac{b^2-r^2}{r^2+b^2}$, according as $b$ is leſs, or greater than $r$. Then will the motion of the ſurface of this ſphere be exactly the ſame as the motion of the ſaid ſpherical ſurface carried about with the revolving ſpheroid

ſpheroid after receiving the impulſe of the percuſſive force. Therefore, having reference to our concluſions in the preceding articles, we, by ſubſtitution, readily obtain ſolution to our problem.

By ſubſtituting properly $\frac{cd}{r} \times \frac{r^2 - b^2}{r^2 + b^2}$ or $\frac{cd}{r} \times \frac{b^2 - r^2}{r^2 + b^2}$ for F, we find,

$$\frac{d^2 r}{\sqrt{d^2 e^2 \mp 2cdr\mathrm{F} + r^2 \mathrm{F}^2}} = \frac{dr}{c} \times \frac{r^2 + b^2}{\sqrt{4r^4 + \overline{r^2 + b^2}|^2 \times \frac{d^2}{c^2}}},$$

$$\frac{r \times \overline{cd \overset{\backsim}{+} r\mathrm{F}}}{\sqrt{d^2 e^2 \mp 2cdr\mathrm{F} + r^2 \mathrm{F}^2}} = \frac{2r^3}{\sqrt{4r^4 + \overline{r^2 + b^2}|^2 \times \frac{d^2}{c^2}}},$$

$$\text{and } c \overset{\backsim}{+} \frac{r\mathrm{F}}{d} = \frac{2r^2 c}{r^2 + b^2}.$$

Which equations, reſpect being had to the concluſions in art. 8. and 9. indicate that, whether $b$ be leſs or greater than $r$, if an immoveable circle DL, whoſe radius is $= \frac{2r^3}{\sqrt{4r^4 + \overline{r^2 + b^2}|^2 \times \frac{d^2}{c^2}}}$, be conceived to be deſcribed in a plane inclined to the plane of the equator of the ſpheroid (before the impulſe) in an angle whoſe ſine (to the radius $r$) is $= \frac{dr}{c} \times \frac{r^2 + b^2}{\sqrt{4r^4 + \overline{r^2 + b^2}|^2 \times \frac{d^2}{c^2}}}$, ſo that the ſaid circle touch the ſaid equator in the point D in the ſection O$p$DQE; the ſpheroid after the impulſe will ſo revolve, that its equator

equator will always touch and roll along the ſaid immoveable circle (DL), the velocity of the point of contact (along that circle) being $= \frac{2r^2c}{r^2+b^2}$, whilſt the ſpheroid turns about its proper axis (OQ) with the primitive angular velocity $c$, and the poles O and Q (by the ſaid rolling of the equator) deſcribe circles (whoſe radii are each $= \frac{bd}{c} \times \frac{r^2+b^2}{\sqrt{4r^4+\overline{r^2+b^2}|^2 \times \frac{d^2}{c^2}}}$) parallel to the ſaid circle DL, with the angular velocity $d$ (or their proper velocity $\frac{bd}{r}$) which we ſuppoſed given to them by the impulſe[(d)].
Thus the motion of the ſpheroid conſequent to the impulſe appears to be remarkably regular.

And in the very ſame manner may be explained the motion of a cylinder, whoſe primitive motion about its proper axis may be diſturbed by ſome percuſſive force in like manner as we ſuppoſed the ſpheroid diſturbed; only (inſtead of the former ſubſtitution for F) ſubſtituting for the accelerative force ariſing from the centrifugal force of the particles of the revolving cylinder its proper value $\frac{cd}{r} \times \frac{3r^2-4b^2}{3r^2+4b^2}$ (computed in our Appendix) and afterwards proceeding as we have done with regard to the ſpheroid,

(*d*) Other ways of ſolving the problem are alſo ſuggeſted by the preceding articles.

*b* de

$b$ denoting half the length of the cylinder, and $r$ the radius of any ſection at right angles to its proper axis.

Seeing that $\left(\frac{cd}{r} \times \frac{3r^2-4b^2}{3r^2+4b^2}\right)$ the expreſſion for the ſaid accelerative force reſpecting the cylinder vaniſhes when $b$ is $= \frac{3^{\frac{1}{2}}r}{2}$, it is manifeſt that the cylinder in that caſe will (with reſpect to its own particles) undiſturbedly revolve about any axis whatever paſſing through its center of gravity, as will a ſphere. Which remarkable property of that particular cylinder I believe has not before been taken notice of.

There are, I am aware, bodies of other forms having the like property.

The preceding articles lead us to conſider the motion of the earth's axis in a light, I preſume, more clear and ſatisfactory than any in which it has before been conſidered; but I muſt, for want of leiſure, defer making the application till ſome future opportunity; only obſerving here, that by what is done above it appears, that from the action of the Sun and Moon on the earth its axis has a diurnal motion, which I have no where ſeen explained. Which motion is not much unlike that of the axis of the revolving ſpheroid juſt now conſidered, when ($2b$) this laſt mentioned axis is many times longer than ($2r$)

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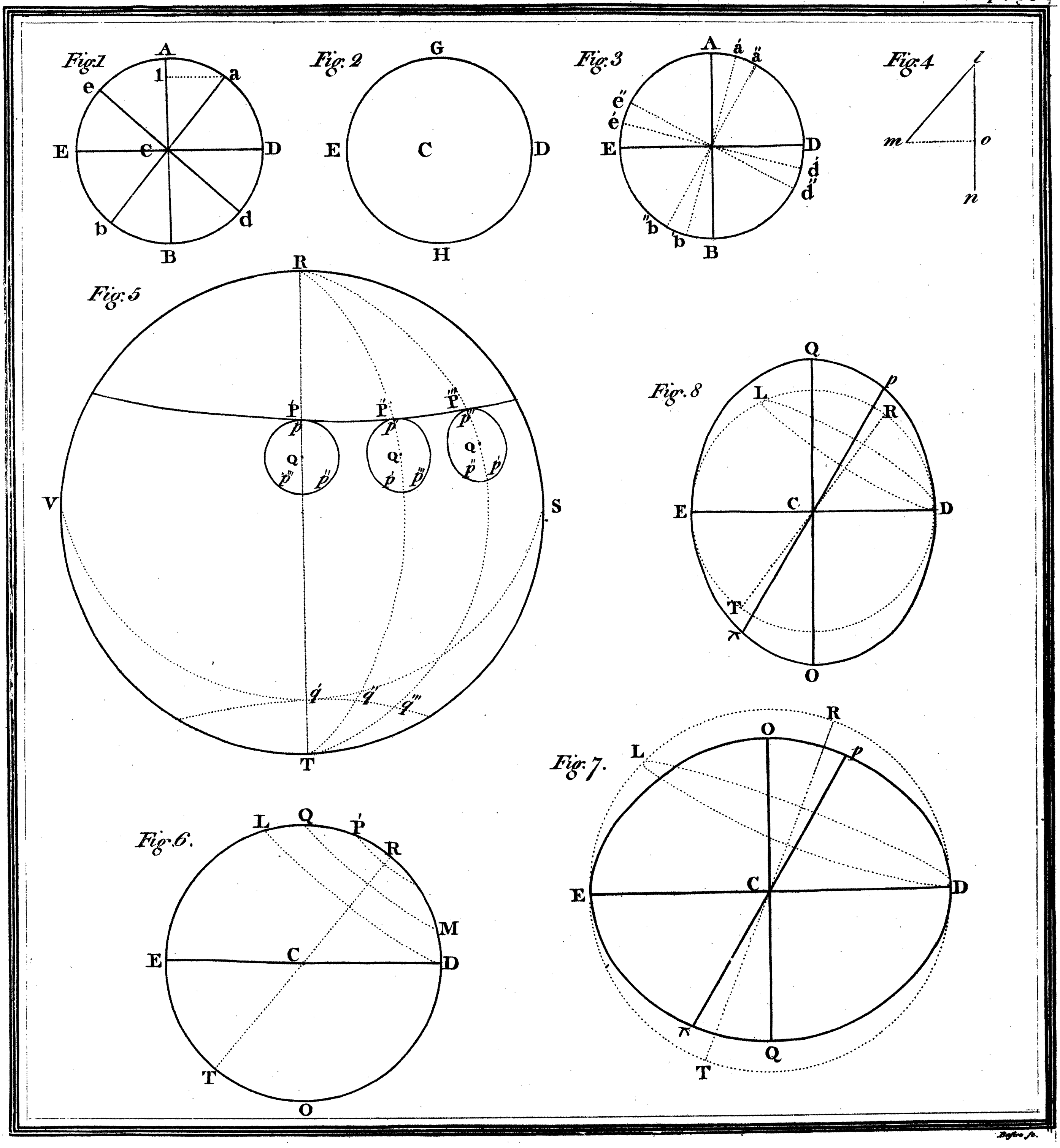
Fig. 1
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Fig. 2
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Fig. 3
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the equatorial diameter of the ſaid ſpheroid, and $\frac{d}{c}$ very ſmall.

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## A P P E N D I X.

### *Shewing how the joint centrifugal force of the particles of a ſpheroid or cylinder, having a rotatory motion about any momentary axis, is computed.*

1. FIG. 9. Let $p$ be a particle of matter firmly connected with the plane DOEFQG, in which the line OCQ is ſituated; and $pq$ being a perpendicular from $p$ to the ſaid plane, let the diſtance $pq$ be denoted by $u$; alſo, the line $ql$ being at right angles to O$l$CQ, let the diſtance $pl$ be denoted by $h$. Then, the ſaid plane with the particle $p$ being made to revolve about O$l$CQ as an axis, with the angular velocity $\dot{e}$ meaſured at the diſtance $a$ from the ſaid axis, the velocity of $p$ will be $= \frac{he}{a}$, and its centrifugal force from $l$ will (by a well-known theorem) be $= \frac{he^2}{a^2}$ to make it $a^2$ the expreſſion being $\frac{he^2}{a^2} \times p$. Whence, by reſolving that force into two others, one in the direction $qp$, and the other in a direction parallel to $lq$, it appears that the force urging $p$ from the plane DOEFQG will be $= \frac{ue^2}{a^2} \times p$, let the diſtance $lq$ be what it will.

 2. The

2. The particle $p$ being connected with the plane DOEFQG as mentioned in the preceding article, and the diſtance $cl$ being denoted by $v$; if $p$ be urged directly from the ſaid plane by a force $fu \times p$, the efficacy of that force to turn the ſaid plane about the line HCI, therein drawn at right angles to OCQ, will (by the property of the lever) be equivalent to the force $\frac{fuv \times p}{g}$ acting on the ſaid line OCQ at right angles to the ſaid plane at the diſtance $g$ from the point C.

Moreover it is obvious, that, *cæteris paribus*, the efficacy will be the ſame let the diſtance of $q$ from $l$ be what it will.

Fig. 10. Let $q$ coincide with $l$; and let $ck$ be a line in the plane $clp$ continued (which plane will be at right angles to the plane DOEFQG); alſo, $pk$ being at right angles to $ck$, let thoſe lines $pk$ and $ck$ be denoted by $w$ and $x$ reſpectively. Then the ſine and coſine of the angle $k$CO to the radius 1, being reſpectively denoted by $m$ and $n$, the force $\frac{fuv \times p}{g}$ will be $= \frac{f \times p}{g} \times \overline{mn \times \overline{w^2 - x^2} + \overline{m^2 - n^2} \times wx}$.

Conſequently, if each particle of any ſolid body, through which a line HCI and a plain DOEIFQGH may be conceived to paſs, be urged from that plane by a force expreſſed by $fu \times p$ as above; the force which, acting on the line OCQ at the diſtance $g$ from C, would be equivalent to the efficacy

efficacy of all the forces acting on the several particles of that body to turn the same about the line HCI will be obtained by computing the sum of all the forces $\frac{f \times p}{g} \times mn \times \overline{w^2 - x^2 + m^2 - n^2 \times wx}$ acting on the said body.

The computation of such equivalent force will in most cases be abridged by observing that, if $pk$ be continued to $\overset{\prime\prime}{p}$ so that $k\overset{\prime\prime}{p}$ be $= kp$, the efficacy of the force on the particle $\overset{\prime\prime}{p}$, to turn the body about the line HCI in opposition to the force on the particle $p$, will be represented by the equivalent force $\frac{f \times \overset{\prime\prime}{p}}{g} \times \overline{mn \times x^2 - w^2 + m^2 - n^2 \times wx}$ acting on the line OCQ at the distance $g$ from C; and that therefore the efficacy of the two forces on $p$ and $\overset{\prime\prime}{p}$, to turn the body about HCI, will be represented by the equivalent force $\frac{2f \times p}{g} \times mn \times \overline{w^2 - x^2}$ acting on the line OCQ, at right angles to the plane DOEIFQGH, at the distance $g$ from C.

3. Fig. 11, 12. If the body be a cylinder, a spheroid, or the like, and its proper axis be situated in the line C$k$, the ordinates corresponding the *abscissæ* $kp$, $k\overset{\prime\prime}{p}$, in the circular section $bi$ whose center is $k$, will each be parallel to that diameter passing through C, about which the body will be urged to turn; and each of those ordinates will be $= \sqrt{y^2 - w^2}$, $y$ being the radius of such section.

 Therefore,

Therefore, writing $2\sqrt{y^2-w^2}$ instead of $p$, it follows that $\frac{4Af}{g} \times mn\dot{x} \times \overline{\frac{y^4}{4} - x^2y^2}$, the whole fluent of $\frac{4f \times \sqrt{y^2-w^2}}{g} \times mn\dot{x} \times \overline{w^2-x^2} \times \dot{w}$, generated $w$ $(=kp=k\ddot{p})$ from o becomes equal to the radius $y$ (both $x$ and $y$ being considered as invariable) will express the value of the force which, acting on the line OCQ at the distance $g$ from C, would be equivalent to the force of all the particles in the said section, whose thickness is denoted by the indefinitely small quantity $\dot{x}$; the distance C$k$ being denoted by $x$, and A being put for (.78539) the area of a quadrant of a circle whose radius is 1.

4. Fig. 11. In the cylinder whose length is $2b$ and diameter $2r$; $y$ being $=r$, $\frac{y^4}{4} - x^2y^2$ will be $= r^2 \times \overline{\frac{r^2}{4} - x^2}$: consequently, the fluent of $\overline{\frac{r^2}{4} - x^2} \times \dot{x}$, generated whilst $x$ from o becomes $= b$, being $\frac{br^2}{4} - \frac{b^3}{3}$, we have $\frac{8Afr^2}{g} \times mn \times \overline{\frac{br^2}{4} - \frac{b^3}{3}} = \frac{fmn}{12g} \times \overline{3r^2 - 4b^2} \times M$ for the force which, acting as above at the distance $g$ from (C) the center of gravity of the cylinder, would be equivalent to the efficacy of the forces acting as above on all the particles of the cylinder to turn it about a diameter passing through C, M being the mass or content of the cylinder.

5. Fig. 12. In the ſpheroid whoſe proper axis is $2b$ and equatorial diameter $2r$, $y^2$ being $= \frac{r^2}{b^2} \times \overline{b^2 - x^2}$, $\frac{y^4}{4} - x^2 y^2$ will be $= r^2 \times \overline{\frac{r^2}{4} - \frac{r^2 x^2}{2b^2} + \frac{r^2 x^4}{4b^4} - x^2 + \frac{x^4}{b^2}}$: conſequently, the fluent of $\frac{r^2 \dot{x}}{4} - \frac{r^2 x^2 \dot{x}}{2b^2} + \frac{r^2 x^4 \dot{x}}{4b^4} - x^2 \dot{x} + \frac{x^4 \dot{x}}{b^2}$, generated whilſt $x$ from o becomes $= b$, being $\frac{r^2 b}{4} - \frac{r^2 b}{6} + \frac{r^2 b}{20} - \frac{b^3}{3} + \frac{b^3}{5} = \frac{2}{15} \times \overline{r^2 b - b^3}$, we have $\frac{16\,\mathrm{A} f r^2}{15 g} \times mn \times \overline{r^2 b - b^3} = \frac{fmn}{5g} \times \overline{r^2 - b^2} \times \mathrm{S}$ for the force which, acting at the diſtance $g$ from C the center of the ſpheroid, would be equivalent to the efficacy of the forces acting as above on all the particles of the ſpheroid to turn it about a diameter of its equator, S being the maſs or content of the ſpheroid.

Theſe equivalent forces are diſtinguiſhed by the name of motive forces; the correſpondent accelerative forces are computed in the following articles.

6. Fig. 13. The body being a ſpheroid whoſe center is C, and whoſe proper axis PN is $= 2b$ and equatorial diameter AB $= 2r$; let F be the accelerative force of a particle at the diſtance $g$ from the axis about which the body is urged to turn, which axis is ſuppoſed to be a diameter of its equator. Denote C$k$ by $x$; $ki$ by $y$; and let the *abſciſſa* $ko$ and its correſpondent ordinate (parallel to the laſt mentioned axis) in the circle whoſe radius is $ki$ be denoted

denoted by $s$ and $t$ respectively. Then, considering the body as urged to turn about that diameter of its equator which is at right angles to AB, the accelerative force of every particle in the said ordinate will be $= \frac{\sqrt{s^2+x^2}}{g} \times F$, and the motive force of all the particles in the same ordinate will be $= \frac{\sqrt{s^2+x^2}}{g} \times F t s = \frac{\sqrt{s^2+x^2}}{g} \times F s \sqrt{y^2-s^2}$; to which (by the property of the lever) a motive force $= \frac{s^2+x^2}{g^2} \times F s \sqrt{y^2-s^2}$ acting at the distance $g$ from the center at right angles to a ray therefrom would be equivalent. Therefore, considering $x$ and $y$ as invariable, and $s$ only as variable, $\frac{4Fx}{g^2} \times$ the whole fluent of $\dot{s}\sqrt{y^2-s^2} \times \overline{s^2+x^2}$ will denote a force which, acting at the distance $g$ from C, would be equivalent to the motive force of all the particles in the section $bi$ whose radius is $ki$ and thickness $x$. Which fluent is $= Ay^2 \times \overline{x^2+\frac{y^2}{4}} = \frac{Ar^2}{b^2} \times \overline{b^2-x^2} \times \overline{x^2+\frac{r}{4b^2} \times \overline{b^2-x^2}}$. Consequently $\frac{8Ar^2F}{b^2g^2} \times$ the whole fluent of $\dot{x} \times \overline{b^2-x^2} \times \overline{x^2+\frac{r^2}{4b^2} \times \overline{b^2-x^2}}$ will denote a motive force which, acting at the distance $g$ from C at right angles to a ray therefrom, would be equivalent to the whole motive force urging the spheroid to turn as above mentioned. Such equivalent

force

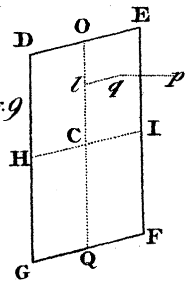


Fig. 9

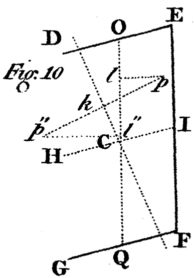


Fig. 10

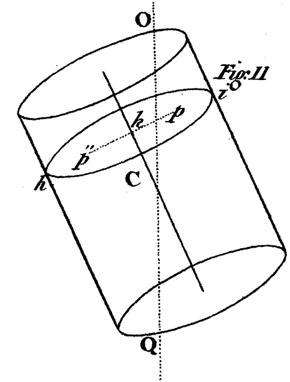


Fig. 11

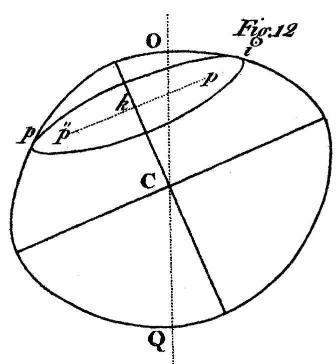


Fig. 12

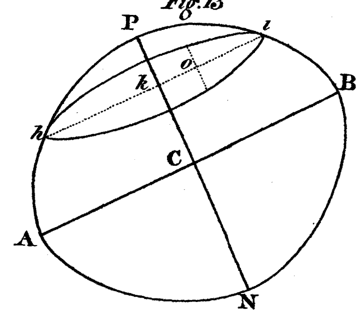


Fig. 13

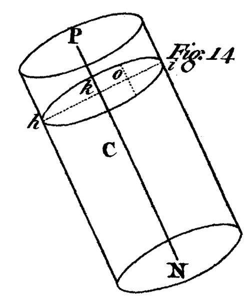


Fig. 14

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force will therefore be $= \frac{16\,A\,r^2\,F}{15g^2} \times \overline{r^4b + r^2b^3} = \frac{F}{5g^2} \times \overline{r^2 + b^2} \times S$; and this being put $= \frac{fmn}{5g} \times \overline{r^2 - b^2} \times S$ (the value of the ſame force found in art. 5.) we find $F = fgmn \times \frac{r^2 - b^2}{r^2 + b^2}$; which will be $= \frac{gmne^2}{a^2} \times \frac{r^2 - b^2}{r^2 + b^2}$, if $f$ be $= \frac{e^2}{a^2}$, its value computed in art. 1.

Or F will be denoted by $\frac{cd}{r} \times \frac{r^2 - b^2}{r^2 + b^2}$; if 1 be to $e$ as $m$ to $d$, and as $n$ to $c$; and $a$ and $g$ be each $= r$.

7. Fig. 14. The body being a cylinder whoſe center of gravity is in C, and whoſe proper axis PN is $2b$ and diameter $2r$; the accelerative force (F) at the diſtance $g$ from C, will in like manner be found $= \frac{gmne^2}{a^2} \times \frac{3r^2 - 4b^2}{3r^2 + 4b^2}$; the cylinder being conſidered as urged to turn about a diameter paſſing through C.

If $1 : e :: m : d :: n : c$, and $a$ and $g$ be each $= r$, F will be $= \frac{cd}{r} \times \frac{3r^2 \quad 4b^2}{3r^2 + 4b^2}$.